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A full-twist inequality for the ν^+ invariant

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1 Back ground

The ν^+ -invariant is a non-negative integer valued knot concordance invariant defined by Hom and Wu [2]. The ν^+ -invariant dominates many concordance invariants derived from Heegaard Floer homology, in terms of obstructions to sliceness, and hence it plays a special role among such knot concordance invariants.

In this section, we give a short review of knot concordance theory and its relationship to Heegaard Floer theory.

1.1 Knot concordance

For two knots K and J in S^3 , let $-K$ denote the orientation reversed mirror image of J and $K\#J$ the connected sum of K and J . We say that K is *concordant* to J if there exists a smooth disk in B^4 with boundary $K\#(-J)$, and we denote the relation by $K \sim_{\text{conc.}} J$. It is well-known that the relation $\sim_{\text{conc.}}$ is an equivalence relation on the set of knots in S^3 , and connected sum endows the quotient set $\mathcal{C} := \{\text{knots in } S^3\} / \sim_{\text{conc.}}$ with an abelian group structure. We often call this group \mathcal{C} the *knot concordance group*.

While the knot concordance group has been studied intensively for more than 50 years, the following fundamental problems are still open.

Problem 1. *Which two knots are concordant?*

Problem 2. *Which knots are concordant to the unknot? (Such knots are called slice knots.) Find an algorithm or algebraic criteria for detecting the sliceness.*

Problem 3. *Determine the group structure of \mathcal{C} . (It is known that \mathcal{C} has $\mathbb{Z}^\infty \oplus (\mathbb{Z}/2\mathbb{Z})^\infty$ as a summand.)*

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To attack these problems, we use many kinds of *knot concordance invariants*, i.e. well-defined maps

$$\mathcal{C} \rightarrow S$$

for some set S .

1.2 Knot concordance invariants from Heegaard Floer theory

Heegaard Floer theory is a Floer homology theory for 3-manifolds established by Ozsváth and Szabó [6, 7]. From Heegaard Floer theory, many knot concordance invariants have been introduced and used to resolve many problems on knot concordance theory. Here we show several such invariants.

- The *correction terms* $d(S_{p/q}^3(-), i) : \mathcal{C} \rightarrow \mathbb{Q}$ ($p/q \in \mathbb{Q}$, $i \in \mathbb{Z}/p\mathbb{Z}$) defined by Ozsváth and Szabó [8]. Originally, these are invariants of Dehn surgeries along a knot.
- The τ -invariant $\tau : \mathcal{C} \rightarrow \mathbb{Z}$ defined by Ozsváth and Szabó [9]. This is famous as a group homomorphism.
- The V_k -sequence $V_k : \mathcal{C} \rightarrow \mathbb{Z}_{\geq 0}$ ($k \in \mathbb{Z}_{\geq 0}$) defined by Ni and Wu [4]. It is known that all correction terms $d(S_{p/q}^3(-), i)$ are determined by the V_k -sequence.
- The ν^+ -invariant $\nu^+ : \mathcal{C} \rightarrow \mathbb{Z}_{\geq 0}$ defined by Hom and Wu [2]. This represents a complexity of the V_k -sequence.
- The *Upsilon invariant* $\Upsilon : \mathcal{C} \rightarrow \text{Cont}([0, 2], \mathbb{R})$ defined by Ozsváth, Stipsicz and Szabó [5]. Here $\text{Cont}([0, 2], \mathbb{R})$ denotes the set of continuous functions on the closed interval $[0, 2]$. This invariant is a group homomorphism whose image contains \mathbb{Z}^∞ as a subgroup.

Then, how strong are these concordance invariants? Actually, they are invariant under a weaker equivalence relation than $\sim_{\text{conc.}}$, which is defined as follows.

Definition 1. For two elements $x, y \in \mathcal{C}$, we say that x is ν^+ -equivalent to y (and denote the relation by $x \sim_{\nu^+} y$) if the equalities $\nu^+(x - y) = \nu^+(y - x) = 0$ hold.

We can verify that the relation \sim_{ν^+} is an equivalence relation, and Hom proves the following theorem.

Theorem 1.1 (Hom [1]). *The quotient $\mathcal{C}_{\nu^+} := \mathcal{C} / \sim_{\nu^+}$ becomes a quotient group of \mathcal{C} . Moreover, the invariants $d(S_{p/q}^3(-), i)$, τ , V_k , ν^+ and Υ are invariant under \sim_{ν^+} . In other words, these invariants can be seen as maps on \mathcal{C}_{ν^+} .*

Theorem 1.1 implies that all the above invariants are determined by the ν^+ -equivalence class of knots. Hence, it is an important problem to understand \sim_{ν^+} and \mathcal{C}_{ν^+} .

Furthermore, ν^+ -equivalence is meaningful for Heegaard Floer theory, too. In [6], Ozsváth and Szabó associated to a knot a “ $\mathbb{Z} \oplus \mathbb{Z}$ -filtered” chain complex CFK^∞ . The $\mathbb{Z} \oplus \mathbb{Z}$ -filtered chain homotopy equivalence class of CFK^∞ is a knot invariant, and we can compute various kinds of Floer homology groups from CFK^∞ ; indeed, we can compute

- the knot Floer homology \widehat{HFK} (and hence we can detect the knot genus and fiberedness as a result), and
- the (all original) Heegaard Floer homology groups $\widehat{HF}, HF^\infty, HF^\pm$ of ALL Dehn surgeries.

In [1], Hom also proves that the ν^+ -equivalence can be translated into an equivalence relation with respect to CFK^∞ . Let $[K]$ denote the concordance class of a knot K .

Theorem 1.2 (Hom [1]). *Two knot concordance classes $[K]$ and $[J]$ are ν^+ -equivalent if and only if there exists a $\mathbb{Z} \oplus \mathbb{Z}$ -filtered chain homotopy equivalence*

$$CFK^\infty(K) \oplus A_1 \simeq CFK^\infty(J) \oplus A_2,$$

where A_1 and A_2 are $\mathbb{Z} \oplus \mathbb{Z}$ -filtered chain complexes with $H_*(A_1) = H_*(A_2) = 0$.

Now, it seems natural to ask the following problems.

Problem 4. *Determine the group structure of \mathcal{C}_{ν^+} .*

Problem 5. *Find geometrical meaning of \sim_{ν^+} .*

In contrast to the case of \mathcal{C} , we can conclude whether a given knot is ν^+ -equivalent to the unknot by using ν^+ . In this work, we mainly consider Problem 5. In particular, we study effects of *full-twist operations* on ν^+ -invariant.

2 Full-twist inequalities for the ν^+ -invariant

As main results of this work, we obtained full-twist inequalities for the ν^+ -invariant. To state the inequalities, we first describe *full-twist operations*. Let K be a knot in S^3 and D a disk in S^3 which intersects K in its interior. By performing (-1) -surgery along ∂D , we obtain a new knot J in S^3 from K . Let $n = \text{lk}(K, \partial D)$. Since reversing the orientation of D does not affect the result, we may assume that $n \geq 0$. Then we say that K is deformed into J by a *positive full-twist with n -linking*, and call such an operation a *full-twist operation*. The main theorem of this paper is stated as follows.

Theorem 2.1. *Suppose that a knot K is deformed into a knot J by a positive full-twist with n -linking. If $n = 0$, then $\nu^+(J\#(-K)) = 0$. Otherwise, we have*

$$\frac{(n-1)(n-2)}{2} \leq \nu^+(J\#(-K)) \leq \frac{n(n-1)}{2}.$$

Remark 1. For any coprime $p, q > 0$, let $T_{p,q}$ denote the (p, q) -torus knot. Then we note that $\nu^+(T_{p,q}) = (p-1)(q-1)/2$ [2, 9], and hence the inequality in Theorem 2.1 implies

$$\nu^+(T_{n,n-1}\#K\#(-K)) \leq \nu^+(J\#(-K)) \leq \nu^+(T_{n,n+1}\#K\#(-K)).$$

Since both $T_{n,n-1}\#K$ and $T_{n,n+1}\#K$ are obtained from K by a positive full-twist with n -linking, the inequalities are best possible for any K .

Here we note that Theorem 2.1 gives an inequality for $J\#(-K)$ rather than J and K . However, by subadditivity of ν^+ , we also have the following result for J and K .

Theorem 2.2. *Suppose that K is deformed into J by a positive full-twist with n -linking. If $n = 0$, then $\nu^+(J) \leq \nu^+(K)$. Otherwise, we have*

$$\frac{(n-1)(n-2)}{2} - \nu^+(-K) \leq \nu^+(J) \leq \frac{n(n-1)}{2} + \nu^+(K).$$

3 Applications

In this section, we show two applications of our full-twist inequalities.

3.1 ν^+ -invariant for cable knots

As an application of Theorem 2.2, we gave a lower bound for the ν^+ -invariant of all cable knots.

Theorem 3.1. *For any knot K and coprime integers p, q with $p > 0$, we have*

$$\nu^+(K_{p,q}) \geq p\nu^+(K) + \frac{(p-1)(q-1)}{2},$$

where $K_{p,q}$ denotes the (p, q) -cable of K .

Note that Wu proves in [10] that the equality holds in the case where $p, q > 0$ and $q \geq (2\nu^+(K) - 1)p - 1$. Hence Theorem 3.1 extends his result to arbitrary cables in the form of inequality. Furthermore, Theorem 3.1 also enables us to extend Wu's 4-ball genus bound for particular positive cable knots to all positive cable knots.

Corollary 3.2. *If $\nu^+(K) = g_4(K)$, then for any coprime $p, q > 0$, we have*

$$\nu^+(K_{p,q}) = g_4(K_{p,q}) = pg_4(K) + \frac{(p-1)(q-1)}{2}.$$

As an application of Corollary 3.2, for instance, we can determine the 4-ball genus for all positive cables of the knot $T_{2,5} \# T_{2,3} \# T_{2,3} \# (-(T_{2,3})_{2,5})$. This example is used in [2] to show that $\nu^+ \neq \tau$. Remark that the τ -invariant cannot determine the 4-ball genus for any positive cable of the knot. Also note that this generalizes [2, Proposition 3.5] and Wu's result in the introduction of [10].

3.2 A partial order on ν^+ -equivalence classes

As another application, we introduced a partial order on \mathcal{C}_{ν^+} and studied its relationship to full-twists by using Theorem 2.1. Our partial order is defined as follows.

Definition 2. For two elements $x, y \in \mathcal{C}_{\nu^+}$, we write $x \leq y$ if $\nu^+(x - y) = 0$.

Note that the equality in the above definition is one of the equalities in the definition of ν^+ -equivalence, and so this partial order seems to be very natural. In fact, we can prove the following proposition.

Proposition 3.3. *The relation \leq is a partial order on \mathcal{C}_{ν^+} with the following properties;*

1. *For elements $x, y, z \in \mathcal{C}_{\nu^+}$, if $x \leq y$, then $x + z \leq y + z$.*
2. *For elements $x, y \in \mathcal{C}_{\nu^+}$, if $x \leq y$, then $-y \leq -x$.*
3. *For coprime integers $p, q > 0$, $k \in \mathbb{Z}_{\geq 0}$ and $0 \leq i \leq p - 1$, all of $-d(S_{p/q}^3(\cdot), i)$, τ , V_k , ν^+ and $-\Upsilon$ preserve the partial order.*

Here the third assertion in Proposition 3.3 implies that there are many algebraic obstructions to one element of \mathcal{C}_{ν^+} being less than another. On the other hand, the following theorem establishes similar obstructions in terms of geometric deformations.

Theorem 3.4. *Suppose that K is deformed into J by a positive full-twist with n -linking.*

1. *If $n = 0$ or 1 , then $[J]_{\nu^+} \leq [K]_{\nu^+}$.*
2. *If $n \geq 3$, then $[J]_{\nu^+} \not\leq [K]_{\nu^+}$. In particular, if the geometric intersection number between K and D is equal to n , then $[J]_{\nu^+} > [K]_{\nu^+}$.*

Here $[K]_{\nu^+}$ denotes the ν^+ -equivalence class of a knot K , and the symbol $>$ means $x \geq y$ and $x \neq y$ for elements $x, y \in \mathcal{C}_{\nu^+}$.

In the above theorem, we can see that only the case of $n = 2$ tells us nothing about the partial order. This follows from the fact that Theorem 2.1 gives $0 \leq \nu^+(x - y) \leq 1$ for $n = 2$ and hence we can show neither $\nu^+(x - y) = 0$ nor $\nu^+(x - y) \neq 0$.

We also mention the relationship between our partial order and satellite knots. Let P be a knot in a standard solid torus $V \subset S^3$ with the longitude l , and K a knot in S^3 . For $n \in \mathbb{Z}$, Let $e_n : V \rightarrow S^3$ be an embedding so that $e(V)$ is a tubular neighborhood of K and $\text{lk}(K, e_n(l)) = n$. Then we call $e_n(P)$ the n -twisted satellite knot of K with pattern P , and denote it by $P(K, n)$. Furthermore, if P represents m times generators of $H_1(V; \mathbb{Z})$ for $m \geq 0$, then we denote $w(P) := m$. It is proved in [3, Theorem B] that the map $[K]_{\nu+} \mapsto [P(K, n)]_{\nu+}$ for any pattern P with $w(P) \neq 0$. We extend their theorem to all satellite knots, and show that those maps preserve our partial order.

Proposition 3.5. *For any pattern P and $n \in \mathbb{Z}$, the map $P_n : \mathcal{C}_{\nu+} \rightarrow \mathcal{C}_{\nu+}$ defined by $P_n([K]_{\nu+}) := [P(K, n)]_{\nu+}$ is well-defined and preserve the partial order \leq .*

By Proposition 3.5, we obtain infinitely many order-preserving maps on $\mathcal{C}_{\nu+}$ which have geometric meaning. Now it is an interesting problem to compare these satellite maps. Theorem 3.4 tells us the relationship among the maps $\{P_n\}_{n \in \mathbb{Z}}$ for some particular patterns.

Corollary 3.6. *Let P be a pattern.*

1. *If $w(P) = 0$ or 1, then the inequality $P_m(x) \geq P_n(x)$ holds for any integers $m < n$ and $x \in \mathcal{C}_{\nu+}$.*
2. *If the geometric intersection number between P and the meridian disk of V is equal to $w(P)$ and $w(P) \geq 3$, then $P_m(x) < P_n(x)$ for any $m < n$ and $x \in \mathcal{C}_{\nu+}$.*

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